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Hilbert Superspaces and Grassmann Numbers

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A heuristic construction of Hilbert superspaces using Grassmann numbers is suggested. Simple applications are given which include a coherent-state formalism for Fermi oscillator systems.

1. INTRODUCTION

The use of anticommuting numbers (ACNs) in physics, introduced by Schwinger (1951) and later extensively considered by Berezin (1966) has become very popular with the development of supersymmetric theories (Wess and Zumino, 1974a, 1974b; Arnowitt, 1975). The invariance groups of such theories, the so-called supergroups, were constructed with the help of ACNs. With the invention of the concept of superspaces (Arnowitt *et al.*, 1975) as the natural setting for supersymmetric theories, the introduction of ACNs in manifold theory was extensively investigated (Berezin and Kac, 1970; Konstant, 1977). In this note we give a heuristic outline of how ACNs may be used in connection with vector-space theory. In Section 2 we briefly review some pertinent properties of Grassmann algebras (GA), and settle our notation and terminology. For the concept of integration in a GA reader is referred to Berezin's book (1966). In Section 3 we use the concept of integration in a GA to define a Hermitian inner product. This step permits a formal construction of Hilbert superspaces (HSS) which we describe in a very heuristic way in Section 4. Simple illustrations are given in Section 5 including the treatment of coherent states for Fermi oscillator systems.

2. GRASSMANN NUMBERS

A graded vector space (over a field[1] K) of elements x of form

$$x = x_0 1 + \sum_i x_i \theta_i + \sum_{i<j} x_{ij} \theta_i \theta_j + \cdots + x_{1\ldots n} \hat{\prod_j} \theta_j \tag{2.1}$$

($\hat{\Pi}_j \theta_j$ denotes the product of the generators θ in the increasing order of j) with coefficients in K is called a GA, $\mathcal{G}$, when equipped with an associative bilinear product which assigns to any pair $(x,y) \in \mathcal{G} \times \mathcal{G}$ a unique element $xy \in \mathcal{G}$, a formula for which is obtained from (2.1) by invoking the basic anticommutation rules[2] (ACR)

$$\theta_i \theta_j + \theta_j \theta_i \equiv \{\theta_i, \theta_j\} = 0 \tag{2.2}$$

The indices i,j, etc. belong to a certain index set I (possibly infinite). The set $G = \{\theta_i | i \in I\}$ is a generator set for $\mathcal{G}$. The number of elements in G is called the degree of $\mathcal{G}$. The set $B = \{1, \theta_i, \theta_i\theta_j \text{ with } i<j, \ldots, \hat{\Pi}_i \theta_i\}$ of products of generators is a basis for $\mathcal{G}$, regarded as a vector space. The number of elements in B is the dimension of $\mathcal{G}$. Clearly, from (2.2) all elements in B, except 1, are nilpotent. Hence a GA is not a division algebra and the cancellation law for multiplication does not hold in general.[3] It is clear from (2.2) that

$$\mathcal{G} = \bigoplus_k \mathcal{G}_k \equiv \bigoplus_{i \in \mathbb{Z}_2} \mathcal{G}^i \tag{2.3}$$

where $\mathbb{Z}_2 = \{0, 1\}$. If $\deg \mathcal{G} = n$, then

$$\dim \mathcal{G}_k = \binom{n}{k} = \frac{n!}{(n-k)!\,k!} \tag{2.4a}$$

and

$$\dim \mathcal{G}^i = 2^{n-1} \tag{2.4b}$$

Each element in $\mathcal{G}_k$ is homogeneous of degree k in the generators. Clearly, $\mathcal{G}_k \mathcal{G}_{k'} \subset \mathcal{G}_{k+k'}$. $\mathcal{G}^i$ consists of sums of monomials of even ($i=0$) or odd ($i=1$) degree. Clearly $\mathcal{G}^i \mathcal{G}^j \subset \mathcal{G}^{|i+j|}$, where $|i+j| \equiv (i+j)$ mod2. The Grassmann parity $\sigma(x^i)$ of $x^i \in \mathcal{G}^i$ is $i \in \mathbb{Z}_2$. From the basic anticommutation rules (2.2) we see that multiplication in $\mathcal{G}$ is $\mathbb{Z}_2$ graded:

$$x^i x^j = (-1)^{|i+j|} x^j x^i, \qquad x^i \in \mathcal{G}^i \tag{2.5}$$

Clearly, from (2.5), the index of nilpotence of an odd element is 2.

[1]We will assume for the most part $K = \mathbb{C}$ (the convenient choice for quantum-mechanical applications) or $K = \mathbb{R}$.

[2]The ACRs (2.2) are limits of the canonical anticommutation rules for fermion systems as Planck's constant tends to zero.

[3]A GA is not an integral domain.

Definition. If $x \in \mathcal{Q}$, the real part $R(x)$ of x is the first coefficient in the polynomial expansion (2.1). If $R(x)=0$, x is called a pure Grassmann and is a null divisor since $x^m = 0$ for some positive integer m. Clearly R has the properties

$$R\left(\sum_{\mu} \alpha_\mu x_\mu\right) = \sum_{\mu} \alpha_\mu R(x_\mu), \forall \alpha_\mu \in K \tag{2.6a}$$

$$R(xy) = R(x)R(y) \tag{2.6b}$$

It is trivial to enlarge a GA: one simply adds elements to the generator set. If $\mathcal{Q}$ and $\mathcal{Q}'$ are two GAs with generator sets G and G', then $G \cup G'$ is a generator set for $\mathcal{Q} \otimes \mathcal{Q}' \equiv \mathcal{Q}\mathcal{Q}' = \{y \mid y = xx',\ x \in \mathcal{Q},\ x' \in \mathcal{Q}'\}$

Definition. Two GAs $\mathcal{Q}$ and $\mathcal{Q}'$ over the same field are isomorphic if there is a one-to-one map between them that preserves (2.5). For given choice of generator sets G, G' there must exist a one-to-one map from G to G'.

Clearly, all GAs of same (finite) degree are isomorphic. If G is a generator set for a finite-dimensional GA $\mathcal{Q}$, over $\mathbb{C}$, then clearly other generator sets can be obtained by means of nonsingular linear maps of G. Thereby we obtain inner automorphisms of $\mathcal{Q}$, under which the real part $R(x)$ of x is trivially seen to be invariant. Thus it is an intrinsic concept.

Definition. Two GAs $\mathcal{Q}, \bar{\mathcal{Q}}$ over the complex field are said to be in conjugation if there is an antilinear isomorphism between them which is in involution, that is, if there exists $C: \mathcal{Q} \to \bar{\mathcal{Q}}$ such that

$$C(x^i x^j) = (-1)^{|i+j|} C(x^i) C(x^j) \tag{2.7a}$$

$$C\left(\sum_{\mu} \alpha_\mu x_\mu\right) = \sum_{\mu} \alpha_\mu^* C(x_\mu) \tag{2.7b}$$

$$C^2(x) = x \tag{2.7c}$$

for every $\alpha_\mu \in \mathbb{C}, x_\mu \in \mathcal{Q}, x^i \in \mathcal{Q}^i$. Notice from (2.7a) that the order of factors is reversed upon conjugation, $C(xy) = C(y)C(x)$. It is especially interesting to consider the direct product of two conjugate GAs $\mathcal{Q}$ and $\bar{\mathcal{Q}}$. Its generator set is $G \cup \bar{G}$, with the anticommutation rules

$$\{\theta_i, \theta_j\} = \{\theta_i, \bar{\theta}_j\} = \{\bar{\theta}_i, \bar{\theta}_j\} = 0 \tag{2.8}$$

Definition. An inner automorphism of a GA which is in involution is called an inner conjugation.

Example. Let $\mathcal{A}\otimes\bar{\mathcal{A}}$ be a GA over $\mathbb{C}$. The map $x\to\bar{x}$ $(x\in\mathcal{A}\otimes\bar{\mathcal{A}})$ defined on $G\cup\bar{G}$ by permutting θ_i with $\bar{\theta}_i$, for every $i\in I$, is an inner conjugation in $\mathcal{A}\otimes\bar{\mathcal{A}}$, obeying

$$\sum_{\mu}\alpha_{\mu}x_{\mu}=\sum_{\mu}\alpha_{\mu}^{*}\bar{x}_{\mu} \qquad \text{and} \qquad R(\bar{x})=R(x)^{*}$$

Theorem. Let $\mathcal{A}$ be a GA over a field K. We assume $\mathcal{A}$ to have a fixed but unspecified degree. The subset $\mathcal{F}$ of nonnilpotent elements in $\mathcal{A}$ is a group.[4]

Proof. Clearly, $\mathcal{F}\equiv\{x|x^{m}\neq0,\ m \text{ any positive integer}\}$ can be characterized as $\mathcal{F}=\{x\in\mathcal{A}|R(x)\neq0\}$. It is clearly closed under multiplication and contains the identity. To show that it is a group write $x\in\mathcal{F}$ as

$$x=\alpha(1+\nu), \qquad \alpha\neq0$$

where $\alpha\in K$ and ν is a pure Grassmann, i.e. $\nu^{m}=0$ for some positive integer m. Define $x_1\equiv\alpha^{-1}(1-\nu)$. Clearly $[x,x_1]=0$ and $xx_1=1-\nu^2$. Now consider $x_2\equiv1+\nu^2$. By associativity, $[x_2,x]=0$ and $(x_1x_2)x=x(x_1x_2)=1-\nu^4$. Now iterate the procedure, always with $[x_j,x]=0$. Clearly, there is a positive integer k for which

$$x'_k x=xx'_k=1, \qquad \text{where } x'_k=\prod_{j=1}^{k}x_j\in\mathcal{F}$$

since ν is nilpotent. Thus x'_k is the inverse x^{-1} of x, and $\mathcal{F}$ is a group. ■

There is a strictly Abelian normal subgroup $\mathcal{F}^{(0)}$ consisting of all nonnilpotent even elements,

$$\mathcal{F}^{(0)}=\{x\in\mathcal{F}|\sigma(x)=0\} \tag{2.9}$$

This fact is relevant for the definition of the inverse of a super matrix (Arnowitt et al., 1975). Finally, from (2.6b) we see that

$$R(x^{-1})=[R(x)]^{-1} \tag{2.10a}$$

which implies

$$R\left(\frac{x}{y}\right)=\frac{R(x)}{R(y)} \qquad \text{all } x,y\in\mathcal{F} \tag{2.10b}$$

Therefore the real part map R establishes a group homomorphism from $\mathcal{F}$ to K. This homomorphism is clearly a projection.

[4]The fact that $\mathcal{A}$ does not possess a true division subalgebra is in accordance with Frobenius theorem: the only division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$ and the real quaternion algebra, $\mathbb{Q}$.

3. GOING FROM GAs TO REAL NUMBERS

If GAs are to be used as underlying structures of physical theories, as it happens in the conventional formulations of supersymmetric theories, a well-defined prescription to associate real numbers to GA elements must be given. Only in this way can such a theory be testable against experiment. That a metric can provide such a rule of association, we show next, using the concept of integration in a GA, as developed in Berezin (1966). Let $x, \bar{y}$ be elements of GAs $\mathscr{G}$ and $\bar{\mathscr{G}}$, conjugate to each other. For given basis, we have

$$x = \sum_{\{\alpha_i\}} x_{\alpha_1 \cdots \alpha_n} \hat{\prod_i} \theta_i^{\alpha_i} \in \mathscr{G} \tag{3.1a}$$

$$\bar{y} = \sum_{\{\alpha_i\}} y^*_{\alpha_1 \cdots \alpha_n} \check{\prod_i} \bar{\theta}_i^{\alpha_i} \in \bar{\mathscr{G}} \qquad \alpha_i \in \mathbb{Z}_2 \tag{3.1b}$$

where the symbol $\hat{\Pi}$ ($\check{\Pi}$) denotes product in the increasing (decreasing) order of the index $i \in I$. The map $\langle \, , \rangle : \mathscr{G} \times \mathscr{G} \to \mathbb{C}$, given by

$$\langle y, x \rangle = \int d\mu(\bar{\theta}\theta) \bar{y}(\bar{\theta}) x(\theta) \tag{3.2}$$

where $d\mu$ is the "Gaussian measure" over anticommuting symbols (Berezin, 1966)

$$d\mu(\bar{\theta}\theta) = \prod_i \exp\left(\bar{\theta}_i \theta_i\right) d\theta_i d\bar{\theta}_i$$

defines the inner product in $\mathscr{G}$. This is the analog for anticommuting symbols of the inner product of the Bargmann–Segal spaces of entire analytic functions [or functionals (Bargmann, 1962; Tabensky and Furtado Valle, 1977)]. Using the fundamental definitions of integration in a GA (Berezin, 1966),

$$\int d\theta_i \theta_i = 1 = \int d\bar{\theta}_i \bar{\theta}_i \tag{3.3a}$$

$$\int d\theta_i = \int d\bar{\theta}_i = 0 \tag{3.3b}$$

we can easily show that (3.2) reduces, for x and y expressed in a given basis, to

$$\langle y, x \rangle = \sum_{\{\alpha_i\}} y^*_{\alpha_i \ldots \alpha_n} x_{\alpha_1 \ldots \alpha_n} \tag{3.4}$$

Since (3.2) is expressed as an integral over $\bar{\mathcal{Q}}\mathcal{Q}$ it is invariant under its inner automorphisms. Furthermore it is clearly a Hermitian inner product, i.e.,

$$\langle y,x\rangle = \langle x,y\rangle^*$$

$$\left\langle y, \sum_\mu \alpha_\mu x_\mu \right\rangle = \sum_\mu \alpha_\mu \langle y, x_\mu\rangle$$

$$\langle x,x\rangle \geqslant 0$$

(equality holds only for $x=0$). The induced norm $\|x\| = \langle x,x\rangle^{1/2}$ clearly obeys triangular and Schwartz inequalities and reduces to $|R(x)|$ in the limit in which x becomes a complex number. Clearly, if a Grassmann element is normalizable, so is its projection onto any homogeneous subspace $\mathcal{Q}_k \subset \mathcal{Q}$.

4. HEURISTIC CONSTRUCTION OF HILBERT SUPERSPACE (HSS)

Let $\mathcal{Q}$ be a GA over $\mathbb{C}$. Clearly, sequences of Grassmann numbers form ordinary (ungraded) vector spaces under usual componentwise addition and multiplication by complex scalars. To combine these vector spaces into $\mathbb{Z}_2$-graded ones we use ACNs in a crucial way. Only products of vectors by Grassmann "scalars" of definite parity are permitted. A vector reverses (keeps) its type under multiplication by an odd (even) Grassmann element. Only addition of vectors of same type is allowed. More precisely we can define what we call the $\mathbb{Z}_2$-graded direct sum $\oplus$ of two vector spaces $\mathcal{H}^i$ over the same field as

$$\mathcal{H} \equiv \bigcup_{i\in\mathbb{Z}_2} \mathcal{H}^i = \bigcup_{i\in\mathbb{Z}_2} \{(\psi^i, \psi^{|i+1|})\}$$

with the operations

(a) $(\psi^i, \psi^{|i+1|}) + (\Phi^i, \Phi^{|i+1|}) = (\psi^i + \Phi^i, \psi^{|i+1|} + \Phi^{|i+1|})$

(b) $\lambda(\psi^i, \psi^{|i+1|}) = (\lambda\psi^i, \lambda\psi^{|i+1|})$

where ψ^i, $\Phi^i \in \mathcal{H}^i$ and λ is either even or odd. The notation $|i+1|$ means $(i+1)$ mod2.

Clearly $\mathcal{H}$ is a group under addition; multiplication by vectors is distributive with respect to scalar addition, and vice versa. Also, multiplication by scalars is associative and there is a unit Grassmann element which reproduces any vector upon multiplication by it. Thus $\mathcal{H}$ is a ($\mathbb{Z}_2$-graded) vector space. We call it a super vector space (SVS). Direct sums and products of SVSs are SVSs. The need for the existence of an underlying field for the construction of a vector space shows itself when one wants to define inverses of operators. However this can be circumvented in the case of SVSs over Grassmann numbers because any GA contains an Abelian group, $\mathcal{F}^{(0)}$, as a subset. First one generalizes the notion of trace to account for the grading and the corresponding generalized definition of determinant (of "supermatrices") follows (Arnowitt et al., 1975). Then, in order for a supermatrix to have an inverse one requires its superdeterminant not to be nilpotent, i.e., to belong to $\mathcal{F}^{(0)}$. Grade-preserving (grade-flipping) operators in $\mathcal{H}$ are defined as those that map $\mathcal{H}^i$ into $\mathcal{H}^j$, where $j=i$ mod2 [$j=(i+1)$ mod2]. One may formally convert an SVS into an HSS by defining a Hermitian inner product with the help of (3.2), which is automatically invariant under GA automorphisms. Under such a product the "even" and "odd" sectors of $\mathcal{H}$ are mutually orthogonal, so that one can confine oneself to one such sector, as long as the relevant operators are grade preserving.

As a realization of a HSS we can consider the set $\mathcal{H}\equiv\cup_i\{\Psi^i\}$ of ordered pairs $\Psi^i=\{\Psi^i(x,\theta),\Psi^{|i+1|}(x,\theta)\}$ under pointwise addition and multiplication by scalars of definite parity ($i\in\mathbb{Z}_2$). The inner product is

$$(\Phi^i,\Psi^i)=\sum_{k=i}^{|i+1|}\int d^nx\int d\mu(\bar{\theta}\theta)\bar{\phi}^k(x,\bar{\theta})\psi^k(x,\theta)\qquad x\in\mathbb{R}^n \tag{4.1}$$

Instead of attempting a systematic study of these structures in general, which may prove a very difficult task, we confine ourselves to simple illustrations.

5. EXAMPLES

(1) Let $\mathcal{Q}_j, j\in I$ (I is an index set, possibly infinite) be a GA generated by θ_j. Let $\mathcal{H}_j$ be complex two-dimensional vector spaces, labeled by $j\in I$, with basis

$$B_j=\{|0\rangle_j,\qquad c_j^\dagger|0\rangle\},\qquad j\in I \tag{5.1}$$

and

$$\{c_j,c_k^\dagger\}=\delta_{jk}\ \{c_j,c_k\}=0=\{c_j^\dagger,c_k^\dagger\} \tag{5.2}$$

We want to develop a coherent-state formalism for Fermi oscillator systems, defined by the Clifford algebra (5.2). The appropriate Hilbert space is $\mathcal{H}=\otimes_{j\in I}\mathcal{H}_j$ and the corresponding GA will be $\mathcal{Q}=\otimes_j\mathcal{Q}_j$, with generator set $G=\{\theta_j|j\in I\}$. We denote by $\bar{\mathcal{Q}}$ its conjugate algebra. Define the state

$$\left|\theta\right\rangle = \exp\sum_j \theta_j c_j^\dagger \left|0\right\rangle \tag{5.3}$$

where the "vacuum" $|0\rangle$ denotes $\otimes_j|0\rangle_j$, $j\in I$ and $|\theta\rangle$ compactly denotes $|\theta_1,\theta_2,\ldots\rangle$. Clearly, in view of (2.2), $|\theta\rangle$ is an eigenstate of the annihilation operator c_j with eigenvalue θ_j,

$$c_j|\theta\rangle = \theta_j|\theta\rangle \tag{5.4}$$

Notice that $c_j, j\in I$, are not grade-preserving operators since they connect "fermionic" states ($c_j^\dagger|0\rangle$) to "bosonic" states ($|0\rangle_j$). Accordingly, they possess odd eigenvalues, equation (5.4). The adjoint of (5.4) gives

$$\langle\theta|c_j^\dagger = \langle\theta|\bar{\theta}_j, \qquad \bar{\theta}_j\in\bar{G} \qquad (\bar{G}\equiv\text{generator set for } \bar{\mathcal{A}}) \tag{5.5}$$

The identity operator in $\mathcal{H}$ is resolved as

$$1 = \int |-\theta\rangle d\mu(\bar{\theta}\theta)\langle\theta| \tag{5.6}$$

with $d\mu(\bar{\theta}\theta) = \Pi_{j\in I}\exp\bar{\theta}_j\theta_j d\theta_j d\bar{\theta}_j$. The set $\{|\theta\rangle\}$ is actually overcomplete since $\langle\theta'|\theta\rangle$ is not zero for $\theta_j \neq \theta_j'$. Instead, $\langle\theta'|\theta\rangle$ is a reproducing kernel $K(\theta',\theta)$ (with respect to $d\mu$) for functions of anticommuting symbols,

$$|\theta\rangle = \int |-\theta'\rangle d\mu(\bar{\theta}'\theta')\langle\theta'|\theta\rangle \tag{5.7}$$

An explicit formula for the kernel is

$$K(\theta',\theta) = \exp\sum_{j\in I}\bar{\theta}_j'\theta_j \equiv \exp\bar{\theta}'\cdot\theta \tag{5.8}$$

Indeed, from (5.6) it is easy to establish the group property for K, namely,

$$K(\theta',\theta) = \int K(\theta',\theta'')d\mu(\bar{\theta}''\theta'')K(\theta'',\theta) \tag{5.9}$$

where each θ,θ',θ'' denotes a set indexed by I.

It is also easy to check that the states defined by (5.3) minimize uncertainty products, by formally duplicating the usual proof for bosonic systems replacing c numbers by ACNs. Define the operators[5]

$$Q_j=\left(\frac{\hbar}{2w_j}\right)^{1/2}\left(c_j^{\dagger}+c_j\right) \tag{5.10}$$

$$P_j=i\left(\frac{\hbar w_j}{2}\right)^{1/2}\left(c_j^{\dagger}-c_j\right)$$

so that

$$\langle\theta|Q_j|\theta'\rangle=\left(\frac{\hbar}{2w_j}\right)^{1/2}\left(\theta_j'+\bar{\theta}_j\right)\exp\bar{\theta}\cdot\theta'=\overline{\langle\theta'|Q_j|\theta\rangle} \tag{5.11a}$$

$$\langle\theta|P_j|\theta'\rangle=i\left(\frac{\hbar w_j}{2}\right)^{1/2}\left(\bar{\theta}_j-\theta_j'\right)\exp\bar{\theta}\cdot\theta'=\overline{\langle\theta'|P_j|\theta\rangle} \tag{5.11b}$$

where, as usual, the bar denotes Grassmann conjugation. The expectation values $\langle\cdots\rangle_{|\theta\rangle}$ of these operators in a (coherent) state $|\theta\rangle$ vanish identically,

$$\langle Q_j\rangle_{|\theta\rangle}\equiv\frac{(\theta,Q_j\theta)}{(\theta,\theta)}=\frac{\int d\mu\langle\theta|Q_j|\theta\rangle}{\int d\mu\langle\theta|\theta\rangle}=0 \tag{5.12}$$

and similarly for P_j, $j\in I$. Furthermore, one finds

$$\langle Q_j^2\rangle_{|\theta\rangle}=\frac{\hbar}{2w_j} \tag{5.13a}$$

and

$$\langle P_j^2\rangle_{|\theta\rangle}=\frac{\hbar w_j}{2} \tag{5.13b}$$

so that the root mean square deviations $\Delta Q_j, \Delta P_j$ of these operators obey

$$\Delta Q_j\Delta P_j=\frac{\hbar}{2},\qquad \text{all } j\in I$$

In a Bargmann wave-function realization of $\mathcal{H}$ the coherent states are given by

$$\langle u|\theta\rangle\equiv\psi_\theta(u)\equiv\exp\theta\cdot u$$

[5]No connection with position or momentum operators is implied.

where $\theta\cdot u=\sum_i \theta_i u_i$, $i\in I$; $\theta_i\in G$, $u_i\in\mathbb{C}$. This is easily seen by noticing that the annihilation operators in this realization are $\partial/\partial u_i$, so that the coherence requirement

$$\frac{\partial}{\partial u_i}\psi_\theta(u)=\theta_i\psi_\theta(u), \qquad \theta_i\in G$$

is trivially satisfied. Furthermore

$$\int d\mu(u^*u)\bar{\psi}_\theta(u^*)\psi_{\theta'}(u)=K(\theta,\theta')$$

where $\bar{\psi}_\theta(u^*)=\exp\bar{\theta}\cdot u^*$ and $d\mu(u^*u)=\exp(-u^*u)(du\,du^*/\pi)$ (in the one-dimensional case). Notice that $K(\theta,\theta')$ is in exact analogy to the corresponding kernel in the bosonic complex variables $u_i, i\in I, \tilde{K}(u,u')=\exp u^*\cdot u'$; algebraic conjugation in the former case plays the role of complex conjugation in the latter. Finally, corresponding expressions may be derived in the ordinary coordinate realization of $\mathcal{H}$, i.e., $\langle x|\theta\rangle\equiv\psi_\theta(x)$.

(2) As a final example we consider the even part $\mathfrak{S}$ of the pseudo-Euclidian HSS used as base-space in supersymmetric theories. Consider a Lorentz 4-vector $z^\mu_{(0)}\in\mathfrak{A}_0\oplus\mathfrak{A}_2$ where $\mathfrak{A}$ is a GA over the reals with fixed but unspecified degree. Adjoin to $z^\mu_{(0)}$ a four-component anticommuting Majorana spinor with components $z^m_{(1)}\in\mathfrak{A}_1$. Clearly, the set $\mathfrak{S}$ of column vectors

$$Z=\begin{pmatrix} z^\mu_{(0)} \\ z^m_{(1)} \end{pmatrix} \tag{5.14}$$

forms a real linear vector space (with the usual operations). The subspace $\mathfrak{S}_L\subset\mathfrak{S}$ of vectors of the form

$$Z_L=\begin{pmatrix} z^\mu_{(0)} \\ 0 \end{pmatrix} \tag{5.15}$$

with $R(z^\mu_{(0)})=z^\mu_{(0)}$ is identified with Minkowski space-time. A real Lorentz-invariant scalar product in $\mathfrak{S}$ is given by

$$(Z',Z)=\int d\mu\left(\hat{z}'^T_{(0)}\eta z_{(0)}+\hat{z}^T_{(1)}Cz_{(1)}\right) \tag{5.16}$$

where we have used matrix notation: η is the Minkowski metric, C the charge conjugation matrix, T denotes transposition, and a hat (^) denotes Grassmann conjugation (here we reserve horizontal bars to denote Dirac adjoints); $d\mu$ is the Gaussian measure in $\mathfrak{A}$.

In $\mathfrak{S}$ we can perform supergauge transformations (Wess and Zumino, 1974a, b) $J:\mathfrak{S}\to\mathfrak{S}$ defined by

$$T(Z)=Z+\delta \tag{5.17}$$

where

$$\delta^{\mu}_{(0)}=\tfrac{1}{2}\bar{z}'_{(1)}\gamma^{\mu}z_{(1)}\in\mathcal{Q}_2$$

$$\delta^{m}_{(1)}=z'^{m}_{(1)}\in\mathcal{Q}_1$$

The supergauge transformation J is labeled by Majorana spinorial anticommuting parameter $z'_{(1)}$ (a suitable real Majorana representation of the Dirac algebra is assumed). It is clear from (5.16) that

$$(Z,\delta)=0 \tag{5.18}$$

since integration with respect to the Gaussian measure will involve an odd number of anticommuting symbols. Therefore a supergauge transformation J may be interpreted as an *orthogonal translation* in $\mathfrak{S}$. Let J_j, $j\in I$ (I a certain index set) denote a family of supergauge transformations. If we perform a succession $S=\Pi_j J_j$ (in a certain unspecified order) of such transformations on a vector $Z_L\in\mathfrak{S}_L$ and furthermore identify $\mathfrak{S}$ with the collection of orbits of Z_L elements under S, then it is clear that $\mathfrak{S}$ admits a decomposition into "longitudinal" and "transverse" components $\mathfrak{S}_L$ and $\mathfrak{S}_T$,

$$\mathfrak{S}=\mathfrak{S}_L\oplus\mathfrak{S}_T \tag{5.19}$$

where $\mathfrak{S}_T=\oplus_{j\in I}\mathfrak{S}^j_T$ contains all nilpotents. As a final remark note that the longitudinal subspace $\mathfrak{S}_L\subset\mathfrak{S}$ is clearly closed under the Poincaré group.

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